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On the Power of R/S-Type Tests under Contiguous and Semi-Long Memory Alternatives

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Abstract. The paper deals with the power and robustness of the R/S type tests under "contiguous" alternatives. We briefly review some long memory models in levels and volatility, and describe the R/S-type tests used to test for the presence of long memory. The empirical power of the tests is investigated when replacing the fractional difference operator $(1 - L)^d$ by the operator $(1 - rL)^d$, with $r < 1$ close to 1, in the FARIMA, LARCH and ARCH time series models. We also investigate the Gegenbauer process with a pole of the spectral density at frequency close to zero.

Mathematics Subject Classifications (2000):

Key words: long memory, Gegenbauer process, ARCH processes, linear ARCH, semi-long memory, modified R/S statistic, KPSS statistic, V/S statistic.

1. Introduction

Recently, much attention has been given to the analysis of long memory time series, see [3, 22]. The hypothesis of long memory was accepted in some data from geophysics, economics, finance, and network traffic. This paper is concerned with the power and robustness of the R/S-type tests under "contiguous" alternatives.

The most popular time series model leading to long memory is fractional ARMA. We analyze two types of fractional time series models. Firstly we consider the fractionally integrated, FI(d), processes characterized by the long memory parameter d. These models can be nested in the class of the Gegenbauer processes, which is characterized by two parameters: the memory parameter d and the frequency $\omega \in [0, \pi)$ of its "persistent component" (see [1, 7, 13]). When $\omega = 0$, this

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process reduces to the FI($2d$) time series. We consider the Gegenbauer process as a "contiguous alternative" for ω in a close neighborhood of zero.

The second class of processes defined using the fractional difference operator $(1-L)^d$ are the long memory ARCH processes, introduced by Robinson [21], and the long memory linear ARCH (LARCH) process introduced by Giraitis, Robinson and Surgailis [10]. We define as contiguous alternatives to these processes, the ones obtained by replacing the fractional difference operator by the mixed operator $(1-rL)^d$ with $r \in (0, 1)$, introduced by Giraitis, Kokoszka and Leipus [8] for the class of ARCH(∞) processes.

The paper is organized as follows. We present in Section 2 the family of fractional ARMA models. The long-range dependent volatility processes are reviewed in Section 3. Section 4 introduces the test statistics. In Section 5 we provide some Monte Carlo results on the power of these tests. The results are summarized in Section 6.

2. Fractional ARMA models

Usually a long memory (LM) process Y_t can be characterized by a single parameter $d \in (0, 1/2)$, called the degree of memory of the process, which controls the shape of the spectrum of the process near the zero frequency and the hyperbolic rate of decay of its autocorrelation function (ACF). More precisely, the spectral density, $f(\lambda)$, of the long memory process is approximated in the neighborhood of the zero frequency by

$$f(\lambda) \sim C\lambda^{-2d} \quad \text{as } \lambda \to 0_+,\ 0 < C < \infty,$$

thus, $f(\lambda) \to \infty$ as $\lambda \to 0_+$. (In the sequel, C stands for generic constant which may change from line to line.) Under additional regularity assumptions of f the ACF $\rho(k)$ of LM process has the following asymptotic behavior:

$$\rho(k) \sim Ck^{2d-1} \quad \text{as } k \to \infty.$$

As a consequence, for $0 < d < 1/2$, $\sum_{k=-\infty}^{\infty} |\rho(k)| = \infty$. This property of absolute nonsummability of autocorrelations is often considered as a definition of long memory and is satisfied by the fractional ARMA models: the fractionally integrated ARMA and the Gegenbauer process. They have an autoregressive (AR) representation:

$$(1 - \alpha(L))Y_t = \varepsilon_t,$$

where ε_t are i.i.d. $N(0, \sigma^2)$ random variables and $\alpha(L) \equiv \sum_{j=1}^{\infty} \alpha_j L^j$ has coefficients with the following rate of decay:

$$\alpha_j = \mathrm{O}(j^{-1-d}) \quad \text{as } j \to \infty.$$

Recall that for the fractionally integrated, FI(d), or FARIMA(0, d, 0) process $\alpha(L) = 1 - (1 - L)^d$, see [12, 14]. The ACF of a FI(d) process and its asymptotics are given by

$$\rho(k) = \frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(d)\Gamma(k+1-d)}, \qquad \rho(k) \sim \frac{\Gamma(1-d)}{\Gamma(d)}\, k^{2d-1} \quad \text{as } k \to \infty,$$

where $\Gamma(\cdot)$ denotes the gamma function. The spectral density of a FI(d) process is equal to

$$f(\lambda) = \frac{\sigma^2}{2\pi}\,(2|\cos\lambda - 1|)^{-d}, \qquad f(\lambda) \sim \frac{\sigma^2}{2\pi}\,\lambda^{-2d} \quad \text{as } \lambda \to 0_+,$$

which means that the FI(d) process is characterized by a "persistent component" at frequency zero. The coefficients of the AR representation of the fractional Gaussian noise FI(d) are given by

$$\alpha_j = \frac{-\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}, \qquad \alpha_j \sim \frac{-1}{\Gamma(-d)}\, j^{-1-d} \quad \text{as } j \to \infty.$$

The Gegenbauer process, suggested by Hosking in the conclusion of [14] and studied independently in [1] and [13], depends on two parameters characterizing a persistent component at frequency ω and the degree of memory d. The AR representation of this process is

$$(1 - 2\cos\omega\, L + L^2)^d Y_t = \varepsilon_t, \quad \omega \in [0, \pi), \ \varepsilon_t \sim N(0, \sigma^2).$$

If $\omega = 0$, this process reduces to a FI($2d$) process. The Gegenbauer process can be interpreted as an AR(∞) process with coefficients

$$\begin{aligned}
\alpha_j &= \sum_{k=0}^{[j/2]} \frac{(-1)^{k+1}\Gamma(-d+j-k)(2\cos\omega)^{j-2k}}{\Gamma(-d)\Gamma(k+1)\Gamma(j-2k+1)},\\
\alpha_j &\sim -\frac{\cos\left((j-d)\omega + (d\pi/2)\right)}{\Gamma(-d)\sin^{-d}(\omega)}\left(\frac{2}{j}\right)^{1+d} \quad \text{as } j \to \infty,
\end{aligned}$$

where $[\cdot]$ indicates integer part. The process has an infinite MA representation:

$$Y_t = \left(1 - 2\cos\omega L + L^2\right)^{-d} \varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \tag{2.1}$$

where the c_j are orthogonal Gegenbauer polynomial coefficients, recursively defined as:

$$\begin{aligned}
&c_0 = 1, \qquad c_1 = 2d\cos\omega,\\
&c_j = 2\cos\omega\left(\frac{d-1}{j} + 1\right)c_{j-1} - \left(2\frac{d-1}{j} + 1\right)c_{j-2}.
\end{aligned}$$

Gray *et al.* [13] have demonstrated the following theorem on stationarity and invertibility of this process:

THEOREM 2.1. *The Gegenbauer process Y_t is stationary if either* $|\cos\omega| < 1$ *and* $d < 1/2$*, or* $|\cos\omega| = 1$ *and* $d < 1/4$*. The Gegenbauer process Y_t is invertible if either* $|\cos\omega| < 1$ *and* $d > -1/2$*, or* $|\cos\omega| = 1$ *and* $d > -1/4$.

The spectral density of a Gegenbauer process is equal to

$$f(\lambda) = \frac{\sigma^2}{2\pi}(2|\cos\lambda - \cos\omega|)^{-2d},$$

$$f(\lambda) \sim \frac{\sigma^2}{2\pi}(2\sin\omega)^{-2d}|\lambda - \omega|^{-2d} \quad \text{as } \lambda \to \omega.$$

Thus, when $d \in (0, 1)$, the spectral density has a pole at frequency ω, which means that this process has a persistent component at frequency ω. The ACF behaves like a cosine wave damped by a hyperbolically decaying sequence:

$$\rho(k) \sim Ck^{2d-1}\cos(k\omega) \quad \text{as } k \to \infty,$$

where the constant C depends on d and ω. Obviously the $\rho(k)$ are summable but their absolute values are not.

The authors of [14] and [12] proposed the generalization of the I(d) processes, fractionally integrated ARMA, or FARIMA(p, d, q), processes defined as

$$\Phi(L)(1-L)^d(Y_t - \mu) = \Theta(L)\varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2),$$

where $\Phi(L)$ and $\Theta(L)$ are the AR and MA lag polynomials of respective orders p and q, which are co-prime on the set of polynomials with real coefficients and with roots outside the unit disk, μ is the unknown mean of the process.

Similarly, Gray *et al.* [13] generalized the Gegenbauer process to the Fractionally Generalized ARMA process, GARMA(p, d, q), defined as

$$\Phi(L)(1 - 2\cos\omega\, L + L^2)^d(Y_t - \mu) = \Theta(L)\varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2).$$

3. Long-range dependent volatility processes

Robinson [21] introduced the class of ARCH(∞) processes defined as:

$$r_t = \sigma_t\varepsilon_t, \qquad \sigma_t^2 = \omega + \psi(L)r_t^2,$$

where $\omega \geqslant 0$, $\psi(L) = \sum_{i=1}^{\infty}\psi_i L^i$ is an infinite order lag polynomial with coefficients ψ_i which are nonnegative and have asymptotically the following hyperbolic rate of decay

$$\psi_j \sim Cj^{-1-d}.$$

Particular parameterizations of the ARCH(∞) process are the LM-ARCH of [6] and the so-called fractionally integrated GARCH (FIGARCH) model of [2]. Both these models involve the fractional difference operator $(1 - L)^d$. In general, the existence of stationary solution for the latter classes have not been theoretically established, see [8, 11, 16]. However, if we replace the fractional difference operator $(1 - L)^d$ by the new operator $(1 - rL)^d$ with $r \in (0, 1)$, i.e. $\psi(L) = (1 - rL)^d$, the modified fractionally integrated GARCH does admit a stationary solution, see [8]. With this operator which mixes hyperbolic and exponential decay, we define a contiguous alternative, called the semi-long memory ARCH, semi LM-ARCH. The ACF of this process satisfies

$$\rho(k) \sim Ck^{d-1}r^k.$$

[23] considered several nonlinear long memory ARCH processes, and their semi-long memory versions. A process of interest is the semi-long memory nonlinear GARCH process defined as

$$\sigma_t^\delta = \omega + \beta\sigma_{t-1}^\delta + \left(1 - \beta L - (1 - \phi L)(1 - rL)^d\right)|\varepsilon_t + \gamma\sigma_t|^\delta.$$

In Table I we present the estimated parameters of a semi LM NGARCH model for the long series of S&P 500 index (period 1929–1994) by using the pseudo maximum likelihood estimator and the assumption that the innovations are t-distributed with η degrees of freedom. We assume that the conditional mean is a MA(1) process accounting for the mild correlation caused by differences in closing times, i.e.,

$$r_t = \mu + \theta\varepsilon_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim t(\eta).$$

Thus, the BIC restrictive criterion would favor the semi-long memory model, although this model has an additional parameter.

Table I. Estimation results (S.E. in parentheses)

Parameters	Estimated parameters	Estimated parameters, with $r = 1$
μ	0.0387(0.0058)	0.0416(0.0058)
θ	0.1409(0.0084)	0.1423(0.0086)
ω	0.0244(0.0058)	0.0305(0.0062)
β	0.3753(0.0744)	0.5676(0.0330)
d	0.2958(0.0299)	0.3719(0.0245)
r	0.9981(0.0005)	1.0000
ϕ	0.2022(0.0559)	0.2364(0.0343)
γ	−0.7102(0.0405)	−0.6038(0.0271)
δ	1.5413(0.0647)	1.3588(0.0498)
η	6.6380(0.3544)	6.5206(0.3467)
BIC	−41890.50	−41959.44

Note that a short memory alternative, but not contiguous, is obtained when $d = 0$. If $\psi(L)$ is a p-order lag-polynomial, the process is a ARCH(p) process, while if $\psi(L)$ is defined as the ratio of two finite order lag polynomials without common roots, the process is a GARCH one.

The long memory linear ARCH, henceforth LM-LARCH, process developed in [10], is defined as

$$r_t = \sigma_t \varepsilon_t, \quad \sigma_t = \alpha + \sum_{j=1}^{\infty} \psi_j r_{t-j}, \tag{3.1}$$

where the ψ_j have the rate of decay $\psi_j \sim Cj^{d-1}$, with $d \in (0, 1/2)$. [10] have shown that under the condition

$$L(E\varepsilon_0^4)^{1/2} \sum_{j=1}^{\infty} \psi_j^2 < 1, \tag{3.2}$$

where $L = 7$ for the Gaussian case and $L = 11$ in other cases, there exist a stationary solution to Equation (3.1), such that the sequence of squares $\{r_t^2\}_{t=1}^{\infty}$ has a covariance function the rate of decay of which is $\rho(k) \sim L(k)k^{2d-1}$ as $k \to \infty$, where $L(k)$ is a slowly varying function. The coefficients ψ_j can be the ones of the MA form of the FARIMA process, i.e.,

$$\psi(L) = 1 + \sum_{j=1}^{\infty} \frac{1 - \theta(L)}{1 - \phi(L)} (1 - L)^{-d}.$$

As before, we define a contiguous short-memory alternative by replacing the fractional difference operator $(1 - L)^d$ by the mixed operator $(1 - rL)^d$. However, unlike for the LM-ARCH case, the limit process when $r = 1$ does exist.

4. *R/S*-Type Tests

[19] proposed a semi-parametric test for the existence of long memory based on Hurst's [15] statistic, with a different normalization for making it more robust to some form of short-range dependence. This statistic is based on the range of the partial sum process $S_k = \sum_{j=1}^{k} (Y_j - \bar{Y}_n)$ and is defined by

$$R/S(q) = \frac{\max_{1 \leqslant k \leqslant n} S_k - \min_{1 \leqslant k \leqslant n} S_k}{\hat{\sigma}(q)},$$

where $\hat{\sigma}(q)$ is defined in (4.1) below, $\bar{Y}_n = n^{-1} \sum_{i=1}^{n} Y_i$. Kwiatkowski *et al.* [17], henceforth KPSS, modified the $R/S(q)$ statistic by replacing the range of the partial sum process by its sample second moment:

$$\text{KPSS}(q) = \frac{1}{n^2 \hat{\sigma}^2(q)} \sum_{k=1}^{n} S_k^2.$$

Originally, that was a test for $I(0)$ against $I(1)$ alternatives. [18] have shown that the KPSS(q) test has power against $I(d)$ process, with $d > 0$.

Giraitis *et al.* [9], have proposed a centering of the KPSS statistic, called the rescaled variance test V/S, based on the sample variance of the partial sum process:

$$V/S(q) = n^{-1}\frac{\widehat{\mathrm{Var}}(S_1, \ldots, S_n)}{\hat{\sigma}^2(q)} = \frac{1}{n^2\hat{\sigma}^2(q)}\left[\sum_{k=1}^{n} S_k^2 - \frac{1}{n}\left(\sum_{k=1}^{n} S_k\right)^2\right].$$

Under the null hypothesis of no long memory, e.g., a $I(0)$ process, short memory linear process, etc., these statistics have well-known asymptotic distributions, which are functionals of the Brownian bridge, $W^0(t) = W(t) - tW(1)$, $W(t)$ being the standard Wiener process.

R/S statistic: $n^{-1/2}\ R/S(q) \Rightarrow \max_{0\leqslant t\leqslant 1} W^0(t) - \min_{0\leqslant t\leqslant 1} W^0(t)$;

KPSS statistic: $\mathrm{KPSS}(q) \Rightarrow \int_0^1 \left(W^0(t)\right)^2 \mathrm{d}t$;

V/S statistic: $V/S(q) \Rightarrow \int_0^1 \left(W^0(t)\right)^2 \mathrm{d}t - \left(\int_0^1 W^0(t)\,\mathrm{d}t\right)^2$.

The distribution functions of these statistics are expressed in the form of series expansions which converge very quickly. Interested readers are referred to Giraitis *et al.* [9] for more details.

The quantity $\hat{\sigma}(q)$ in the above statistics is defined by

$$\hat{\sigma}^2(q) = \hat{\gamma}_0 + 2\sum_{i=1}^{q} \omega_i(q)\hat{\gamma}_i, \quad \omega_i(q) = 1 - \frac{1}{q+1}, \tag{4.1}$$

which is the HAC variance estimator proposed in [20], the sample autocovariances $\hat{\gamma}_i$ at lag i account for the possible short-range dependence up to the qth order, with weights $\omega_i(q)$ corresponding to the Bartlett window. There is no selection rule for choosing the order q, although q should be related to the sample size to satisfy $1/q + q/n \to 0$ as $n \to \infty$. A standard choice is $q = \sqrt{n}$. However, given that the power of the tests tend to their size for that bandwidth, we consider alternative selection rules such as the VARHAC estimator $\hat{\sigma}_V^2$, introduced in a multivariate framework in [5]. The estimator $\hat{\sigma}_V^2$, is defined as

$$\hat{\sigma}_V^2 = \frac{\hat{\sigma}_k^2}{(1 - \sum_{i=1}^{k} \hat{\zeta}_i)^2},$$

where $\hat{\sigma}_k^2$ is the innovation covariance matrix of an AR model fitted on the series, $\hat{\zeta}_i$ are the estimated coefficients of the AR model, the order k of which is selected by using either the Akaike (AIC) or the Schwarz (BIC) information criteria. We denote these estimators by $\hat{\sigma}_{V_{\mathrm{BIC}}}^2$ and $\hat{\sigma}_{V_{\mathrm{AIC}}}^2$.

Since described tests are based on the typical shape of the spectrum near the zero frequency of long memory series, they work well for the parametric class of FI(d) processes, but are by construction invalid with the Gegenbauer process if $\omega > 0$, i.e., the process has a persistent component at a frequency strictly larger than zero. Our purpose is to investigate the power of these tests against contiguous alternatives: long memory models with a singularity at a frequency in a very close positive neighborhood of the zero frequency, and the semi-long memory FARIMA, LARCH, and ARCH, in which the fractional difference operator $(1-L)^d$ is replaced by the operator $(1-rL)^d$ with $r < 1$.

5. Monte Carlo Simulations

We consider the following data generating processes (DGP), which correspond to the "almost" long memory alternatives. The null hypotheses are defined by setting $d = 0$ for DGP A, B and C, and by DGP E for DGP D.

DGP A: the Gegenbauer process,

$$(1 - 2\cos\omega L + L^2)^d Y_t = \varepsilon_t, \quad \varepsilon_t \sim N(0, 1),$$

for several values of d and ω, i.e., d = 0.10, 0.15, 0.20, with $\cos\omega$ = 0.99999, 0.9999, 0.999, 0.99, 0.95, i.e., ω = 0.00447, 0.01414, 0.04472, 0.14154, 0.31756.

We also consider the cases of poles of the form $\omega = cn^{-1/2}$, for n = 100, 200, ..., 1000, and $c = 1$.

DGP B: the semi-long memory FARIMA(0, d, 0) process,

$$(1 - rL)^d Y_t = \varepsilon_t, \quad \varepsilon_t \sim N(0, 1),$$

for several values of r = 0.999, 0.99, 0.9 and d = 0.20, 0.30, 0.40.

DGP C: a semi-long memory LARCH process,

$$r_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1), \ \sigma_t = \alpha + \sum_{j=1}^{\infty} \psi_j r_{t-j},$$

where the coefficients ψ_j are the ones of the MA form of the FARIMA(1, d, 1) process, where $(1 - L)^d$ is replaced by $(1 - rL)^d$, i.e.

$$\psi(L) = 1 + \sum_{j=1}^{\infty} \frac{1 - \theta L}{1 - \phi L}(1 - rL)^{-d},$$

for several values of r = 0.999, 0.99, 0.9 and d = 0.20, 0.30, 0.40.

DGP D: a semi-long memory ARCH process,

$$r_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1), \ \sigma_t^2 = \alpha + \sum_{j=1}^{\infty} \psi_j r_{t-j}^2,$$

where the coefficients ψ_j are the ones of a FIGARCH process, in which the filter $(1-L)^d$ is replaced by $(1-rL)^d$, for several values of $r = 0.999, 0.99, 0.9$ and $d = 0.20, 0.30, 0.40$.

DGP E: We consider three particular cases of short memory alternatives for DGP D: an ARCH(1), an ARCH(2), and a GARCH(1, 1) process.

We considered here two sample sizes: $n = 500, 1000$. The expansion of fractional filters is truncated at the order 2000. DGP A and DGP B use 2000 pre-sample observations, i.e. the length of the fractional filter since these DGP have a MA form. However, since DGP C and DGP D do not have such a MA form, we use 30000 pre-sample observations for avoiding dependence on initial conditions.

Given that the three statistics R/S, V/S and KPSS converge to random variables with known analytical distribution functions, we can compute the P-values for each simulation result. These statistics are computed for the HAC estimator $\hat{\sigma}^2(q)$ and several values of the autocorrelation order $q = 0, 1, 2, 5, 10, 20, 30$, and for the estimators $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$ and $\hat{\sigma}^2_{V_{\mathrm{AIC}}}$.

6. Results

Given that we are considering a large number of DGP's, presenting the whole set of results with tables would be space consuming. Thus, we summarize some of the results by using the size-power curves advocated in [4], which is the plot of the empirical distribution, henceforth EDF, of the P-values of the DGP under the alternative hypothesis against the EDF of the P-values of the DGP satisfying the null hypothesis. (The whole set of results is available upon request.)

From the whole set of results, we conclude that the R/S statistic is more sensitive than the KPSS and the V/S statistic to the choice of the variance estimator $\hat{\sigma}^2$. For the ARCH(p) and GARCH(1, 1) null hypotheses, unlike the KPSS and the V/S tests, this test does not have the correct size when using the $\hat{\sigma}^2_{V_{\mathrm{AIC}}}$ or $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$ estimators. This sensitivity casts doubts on the existence of a reliable and robust selection rule for the denominator of the R/S statistic.

For the semi-long memory LARCH, FARIMA, FIGARCH, the results do not differ too much from the standard long memory case when $r = 0.999$. However, when r is smaller, the power of the three tests declines, although for the semi-long memory FARIMA process, for $d = 0.20, 0.30, 0.40$, and $r = 0.9$, the V/S and R/S test do have some power for $q = 0, 1$, but no power at all for larger values of q and for both estimators $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$ and $\hat{\sigma}^2_{V_{\mathrm{AIC}}}$, see Figure 1.

The three tests have less power for the semi-long memory volatility models. For the LARCH case, if condition (3.2) is not satisfied, the test has more power for small values of d, i.e., $d = 0.20$, the power increases for $d = 0.30$ and decreases for $d = 0.40$. We meet here the issue of nonmonotonicity of the power function when the process is not stationary or close to non stationarity, and the statistic of interest depends on an estimator of the variance, see [24] and [9] for further references.

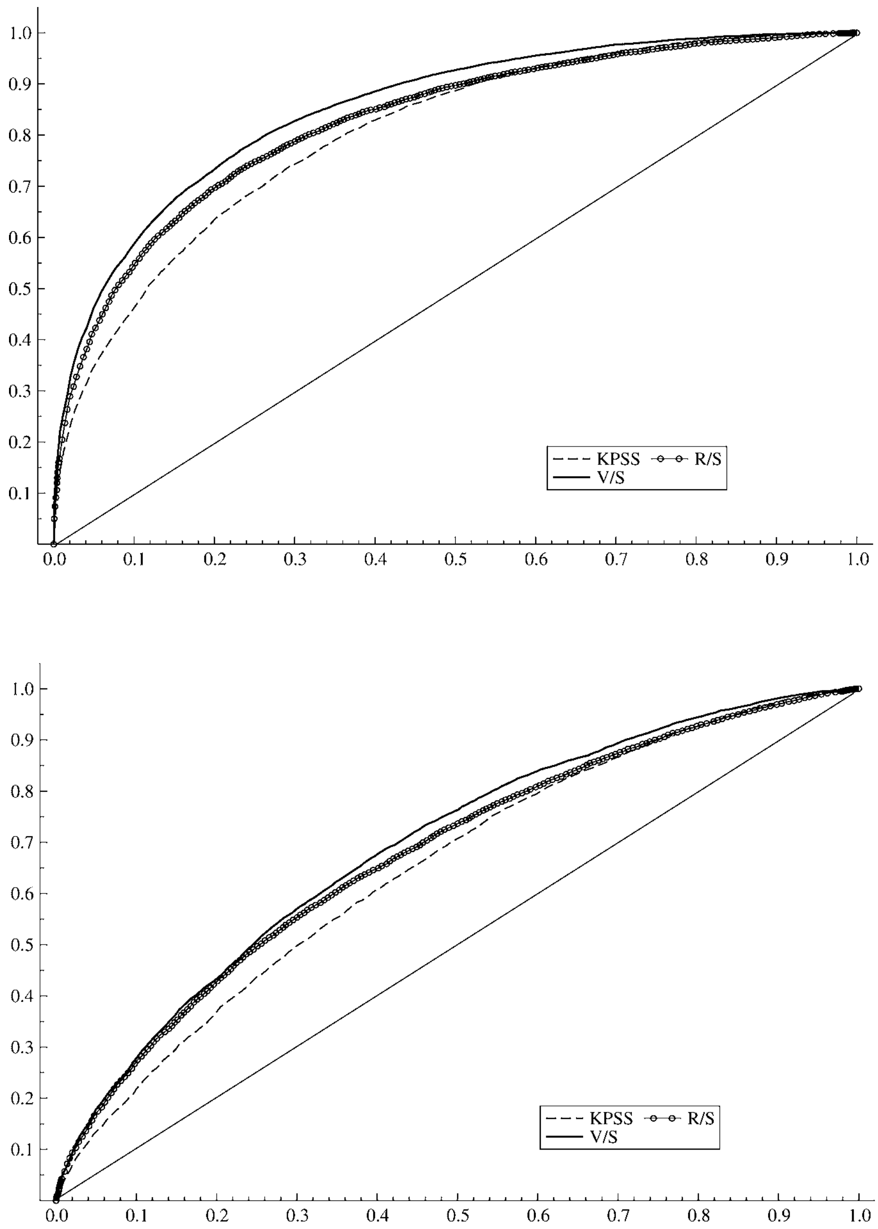

Figure 1. Semi FARIMA process: $d = 0.30$, $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$, $n = 500$, $r = 0.99$ (top) and $r = 0.9$ (bottom).

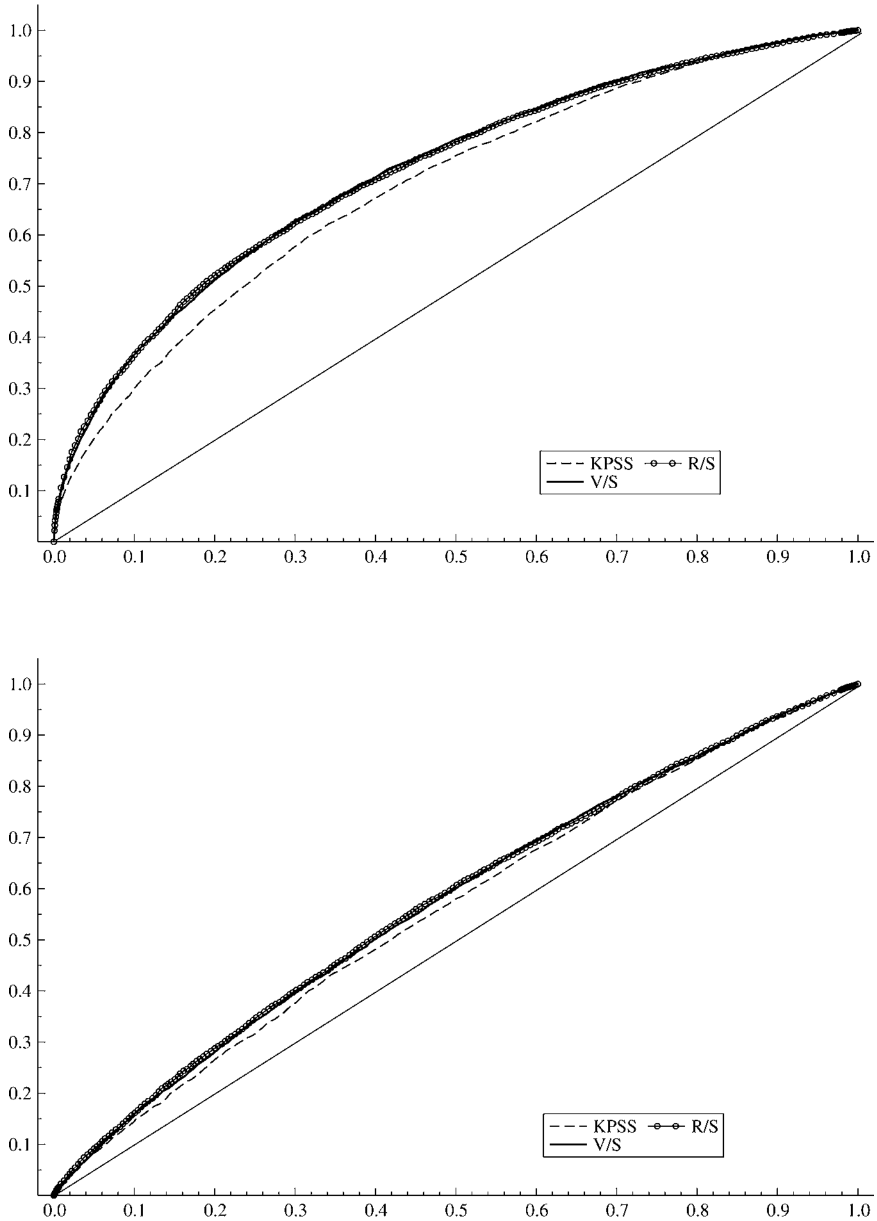

Figure 2. Semi LARCH(1, d, 1) process: $\phi = 0.1$, $\theta = 0.2$, $d = 0.30$, $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$, $n = 500$, $r = 0.99$ (top) and $r = 0.9$ (bottom).

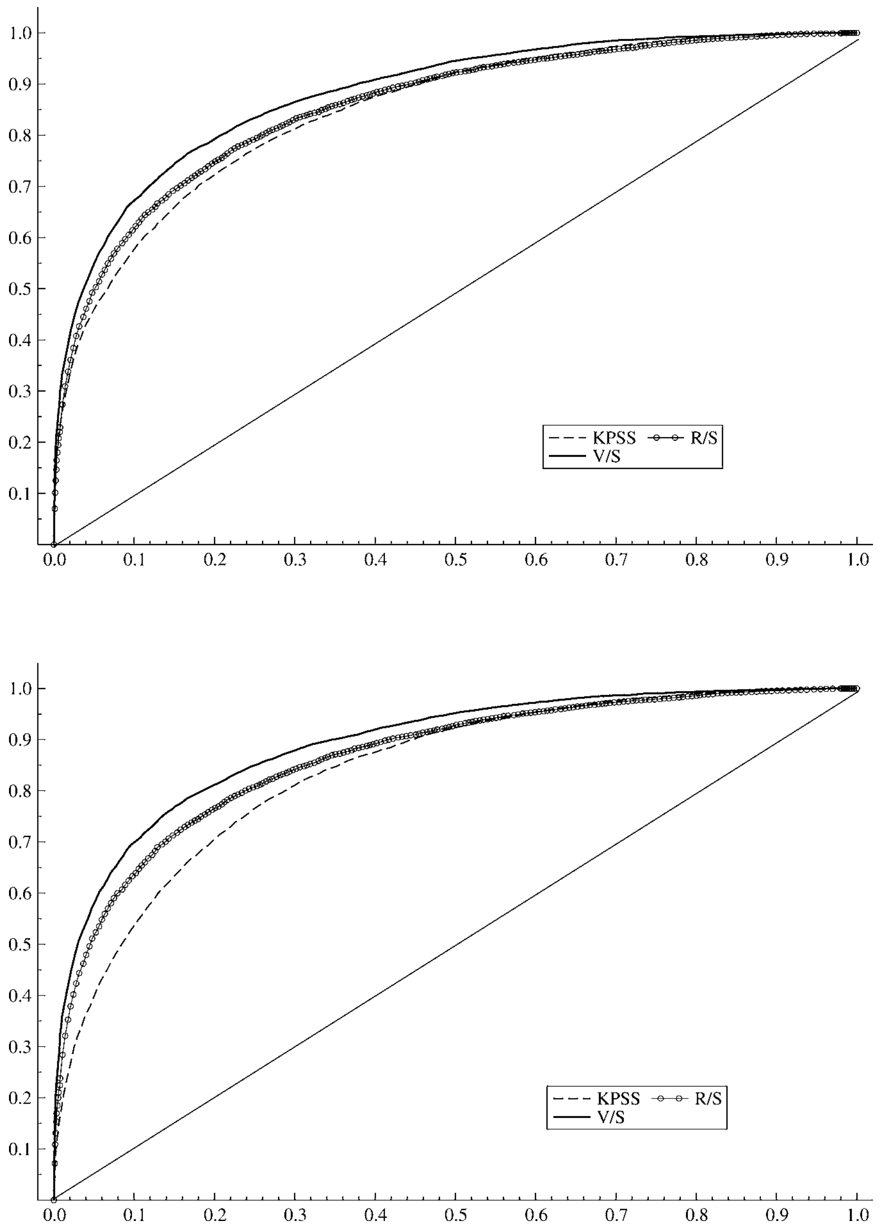

Figure 3. Gegenbauer process: $d = 0.15$, $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$, $n = 500$, $\cos\omega = 0.99999$ (top) and $\cos\omega = 0.9999$ (bottom).

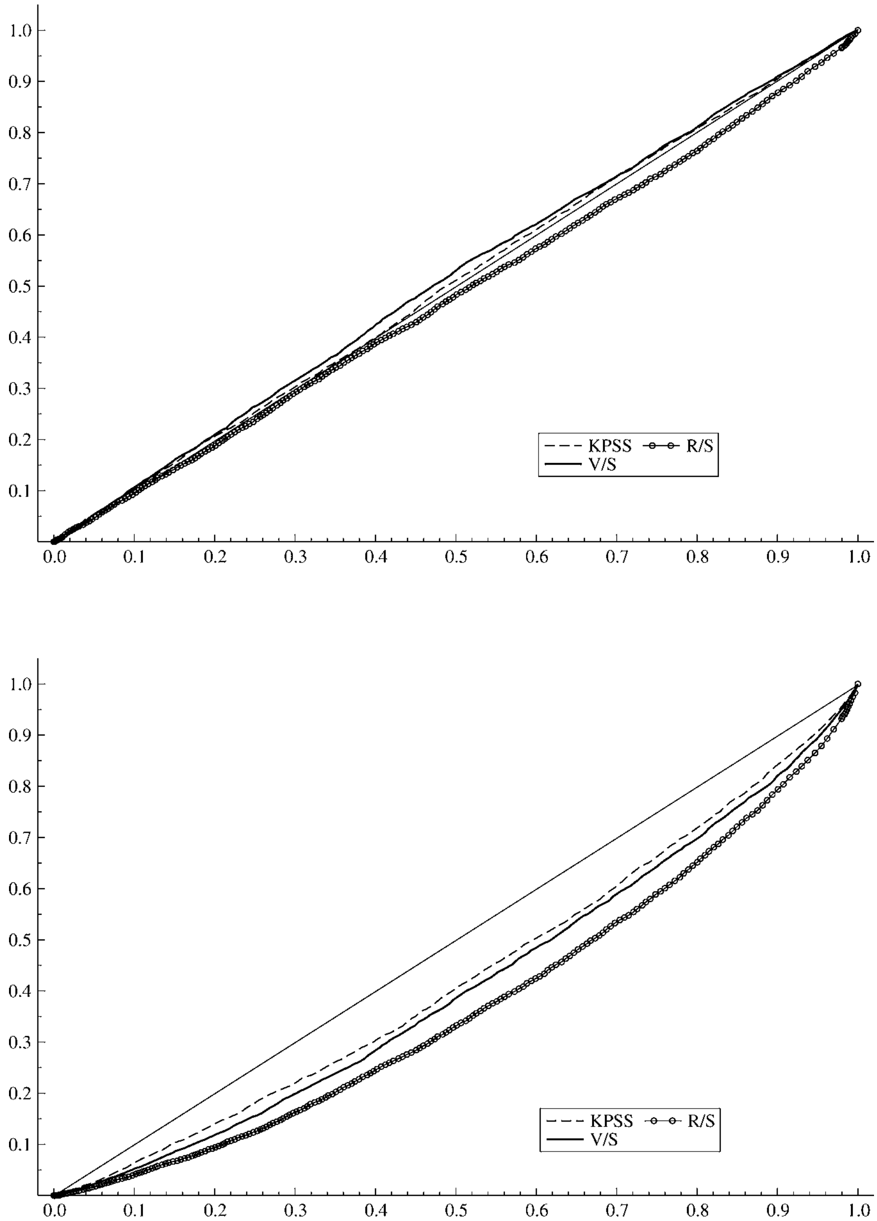

Figure 4. Gegenbauer process: $d = 0.15$, $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$, $n = 500$, $\cos\omega = 0.99$ (top) and $\cos\omega = 0.95$ (bottom).

For the LARCH case, with condition (3.2) satisfied, the three tests do have little power for $d = 0.20$. However, these tests detect the presence of long memory when $d \geqslant 0.30$, for large samples, i.e., $n = 1000$, and $r \geqslant 0.99$. For $r = 0.9$, these tests have little power, see Figure 2, even for $d = 0.40$.

Results are much more appealing for the semi-long memory FIGARCH, as all the tests are able to detect the presence of long range dependence, even for $r = 0.9$, provided that q is not too large. In fact, for this class of process, using either estimators $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$ or $\hat{\sigma}^2_{V_{\mathrm{AIC}}}$ is not appropriate as the AIC and BIC criteria would select a too large number of lags. The statistics appear to be more sensitive to q for this class of processes than for the semi-long memory FARIMA process. This result is of practical interest if we wish to detect the presence of long range dependence in the series of squared returns of asset prices, and if we believe that the appropriate model is a (semi) long memory ARCH, as seen in Section 3. In that case, we should select q small, i.e., $q \leqslant 5$, for the HAC estimator.

For the Gegenbauer processes when $\cos\omega \leqslant 0.95$, i.e., $\omega \geqslant 0.31756$, the KPSS and V/S statistics do not reliably detect the presence of long memory, the R/S statistic has some power, however when q increases for the HAC estimator, and for the estimator $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$, the R/S statistic becomes biased as well, i.e., its power is lower than its size. However, when the poles are close to the zero frequency, $\omega \leqslant 0.04472$, and for $q < 5$, the three tests detect the presence of long memory. For the case of poles of the form $\omega = cn^{-1/2}$, the tests detect the presence of long memory for $n = 100$, but for $q < 5$. When the sample size increases, the tests have obviously more power as the singularity becomes closer to zero.

For the LARCH case, we assume that condition (3.2) is satisfied.

The curves obtained with the HAC estimator $\sigma^2(q)$ for q small, i.e., $q = 0, 2, 5$ are over these curves obtained with the $\hat{\sigma}^2_{V_{\mathrm{BIC}}}$ estimator. We consider small samples, $n = 500$ and $d = 0.30$. For all DGP, the three statistics detect the presence of long memory in the simulated series, except for the Gegenbauer process when $\cos\omega = 0.99$, see Figure 4, in that case the power of the tests are equal to their size. When $\cos\omega = 0.95$ (see Figure 4), the test is biased; even for the HAC estimator with $q = 2$, the size power curve is slightly over the 45° line.

We also observe that for the figures reported here, the V/S test has more power than the R/S test. This fact, which results from the lower sensitivity of the V/S statistics to the variance estimator, advocates the use of this statistic.

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