Double Long-Memory Time Series

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Introduction

Purpose of the paper:

- Present a new class of stochastic processes which surimpose a long-range dependent process in the conditional mean and a long-range dependent process in the conditional variance
- Propose an estimation procedure based on the maximization of an approximate log-likelihood function.

Long–memory in conditional mean:

A stationary process $\{Y_t\}$ is called a stationary process with long-memory if its autocorrelation function (ACF) has asymptotically the following hyperbolic rate of decay:

$$\text{Cov}(Y_k, Y_0) \sim L(k)k^{2d-1} \quad \text{as } k \to \infty$$

where

- $L(k)$ is a slowly varying function,[a]
- $d \in (0, 1/2)$ is the parameter which governs the slow rate of decay of the ACF.
- d characterizes the degree of long-memory, or the persistence of the series

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[a] A function $L(k)$, $k \geq 0$, is called slowly varying function if $L(\lambda k)/L(k) \to 1$ as $k \to \infty$, $\forall \lambda > 0$.

Exemple of long-range dependent process:
ARFIMA (p, d, q)

$$(1 - L)^d y_t = z_t$$

where z_t is a stationary and invertible short-memory ARMA(p, q) process. and

$$\begin{aligned} (1 - L)^d &= \sum_{k=0}^{\infty} \psi_k L^k, \\ \text{with} \quad \psi_0 &= 1, \ \psi_k = \prod_{j=1}^{k} \left(1 - \frac{1+d}{j}\right) \end{aligned}$$

where the infinite sequence of coefficients $\{\psi_j\}_{j=1}^{\infty}$ has an asymptotic hyperbolic rate of decay controlled by the fractional degree of integration d:

$|\psi_j| \sim Aj^{-(1+d)}$ as $j \to \infty$, A is a constant > 0

Long–memory in conditional variance:

Robinson (1991) introduced the class of long-memory ARCH processes:

$$
\begin{aligned}
r_t &= \mu + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \mathrm{N}(0,1), \\
\sigma_t^2 &= \sigma^2 + \sum_{i=1}^{\infty} \psi_i (\varepsilon_{t-i}^2 - \sigma^2) = \sum_{i=1}^{\infty} \psi_i \varepsilon_{t-i}^2
\end{aligned}
$$

for some $\sigma^2 > 0$, with

$$\psi_j = O\left(j^{-(1+d_\varepsilon)}\right)$$

slow (hyperbolic) rate of decay.

General form

$$
\begin{aligned}
r_t &= \mu + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \mathrm{D}(0,1), \\
\sigma_t^\delta &= \omega + \psi(\Theta_\psi, d_\varepsilon)(L) g(\varepsilon_t; \Theta_g) \\
\text{with} \quad & E\left(\sigma_t^\delta - g(\varepsilon_t; \Theta_g)\right) = 0
\end{aligned}
$$

FIGARCH(l, d_ε, m) :

$$\psi(\Theta_\psi, d_\varepsilon)(L) = 1 - \phi(L)\,(1-\beta(L))^{-1}\,(1-L)^{d_\varepsilon},$$

$$\sigma_t^\delta = \frac{\omega}{1-\beta(1)} + \left(1 - \frac{\phi(L)(1-L)^{d_\varepsilon}}{1-\beta(L)}\right)|\varepsilon_t|^\delta$$

Long-memory ARCH(d_ε)

$$\sigma_t = \left(1-(1-L)^{d_\varepsilon}\right)\frac{|r_t|}{\lambda} \quad \text{where} \quad \lambda = E\,|\varepsilon_t|$$

LM–ARCH(p, q) : $\psi(\Theta_\psi, d_\varepsilon)(L) =$
$\sum_{j=1}^{\infty} B(p+j-1, d_\varepsilon+1)L^j / B(p, d_\varepsilon)$, and
$g(\varepsilon_t; \Theta_g) = |\varepsilon_t|^\delta$,

$$\sigma_t^\delta = \sum_{j=1}^{\infty} \frac{B(p+j-1, d_\varepsilon+1)}{B(p, d_\varepsilon)} |\varepsilon_{t-j}|^\delta$$

Combining long memory models in conditional mean and conditional variance:

Double long-memory models

ARFIMA-FIGARCH

ARFIMA$(1, d_y, 1)$-FIGARCH $(1, d_\varepsilon, 1)$ process

$$(1-\psi_1 L)(1-L)^{d_y}(y_t-\mu_y) = (1+\theta_1 L)\,\varepsilon_t,$$

$$\varepsilon_t \sim \text{i.i.d } \mathrm{N}(0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \left(1-\beta_1 L - (1-\phi_1 L)(1-L)^{d_\varepsilon}\right)\varepsilon_t^2$$

ARFIMA-LM-ARCH

ARFIMA$(1, d_y, 1)$-LM-ARCH (p, d_ε) process

$$(1-\psi_1 L)(1-L)^{d_y}(y_t-\mu_y) = (1+\theta_1 L)\,\varepsilon_t,$$

$$\varepsilon_t \sim \text{i.i.d } \mathrm{N}(0, \sigma_t^2)$$

$$\sigma_t^\delta = \sum_{j=1}^{\infty} \frac{B(p+j-1, d_\varepsilon+1)}{B(p, d_\varepsilon)} |\varepsilon_{t-j}|^\delta$$

Practical interest

Some time series display a long-memory component in the conditional mean and in the conditional variance

Exemples:

- Absolute daily returns and squared daily returns $|R_t|^\delta = |\ln(P_t/P_{t-1})|^\delta$.
- Intraday volatility defined as

$$\hat{\sigma}_t^2 = \frac{(\ln H_t - \ln L_t)^2}{4 \ln 2}$$

- Intraday returns, e.g., based on 20 minutes spaced observations. (See Olsen & Associates data). Intraday returns are antipersistent, i.e., $d_y < 0$.

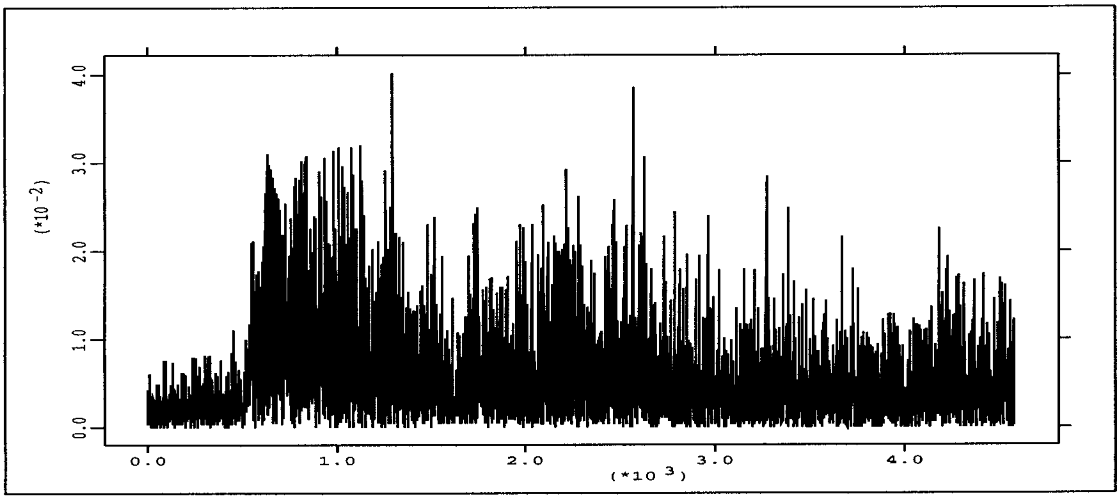
Some remarks

- In fact, the power transformations of returns and intraday volatility share the second moment properties of long-memory processes, i.e., a hyperbolic rate of decay of the autocorrelation function:

$$\text{Cov}(R_t^2, R_{t+k}^2) \sim C\, k^{2q-1} \quad \text{as} \quad k \to \infty$$

- However, unlike long-range dependant processes, the absolute returns and intraday volatility series do not exhibit a trend.
- For that reason, we termed this series as pseudo long-memory processes

Figure 1: Absolute returns on T-bill futures



The degree of long-memory is around $d = 0.45$ but the series does not exhibit a trend.

Double long-memory models:

- Estimation of the parameters by minimizing an approximate log-likelihood function

$$\mathcal{L}_n(\zeta) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{n}\left(\ln \sigma_t^2 + \frac{\varepsilon_t^2}{\sigma_t^2}\right)$$

- Diagnostic checking: Ljung-Box statistic based on
  - standardized residuals $\widetilde{\varepsilon}_t = \widehat{\varepsilon}_t/\widehat{\sigma}_t$
  - absolute standardized residuals $|\widehat{\varepsilon}_t/\widehat{\sigma}_t|$
  - squared residuals $\widehat{\varepsilon}_t^2/\widehat{\sigma}_t^2$

$$Q_K = n(n+2)\sum_{i=1}^{K}\frac{\widehat{\rho}_i\left(\widetilde{\varepsilon}\right)}{n-i}$$

Under H_0 $Q_K \sim \chi^2(l)$
For an ARIMA(p,d,q) model, the number of degrees of freedom $l = K - p - q$

Figure 2: Density of $\widehat{d_y}$, with 4500 observations

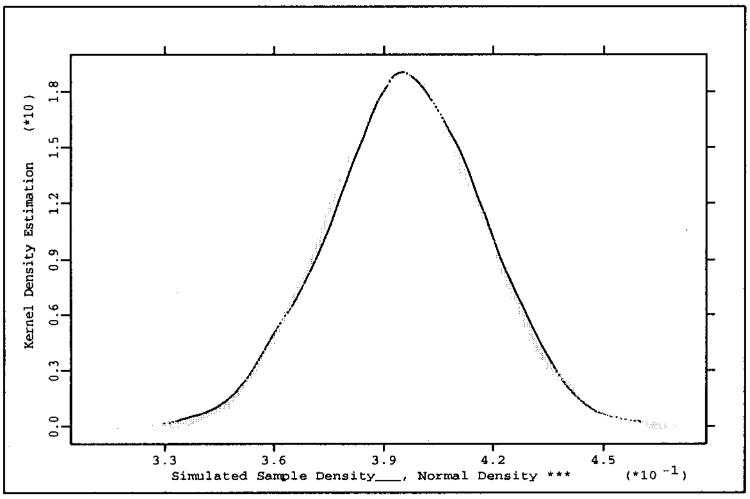


Figure 3: Density of $\widehat{d_y}$, with 8500 observations

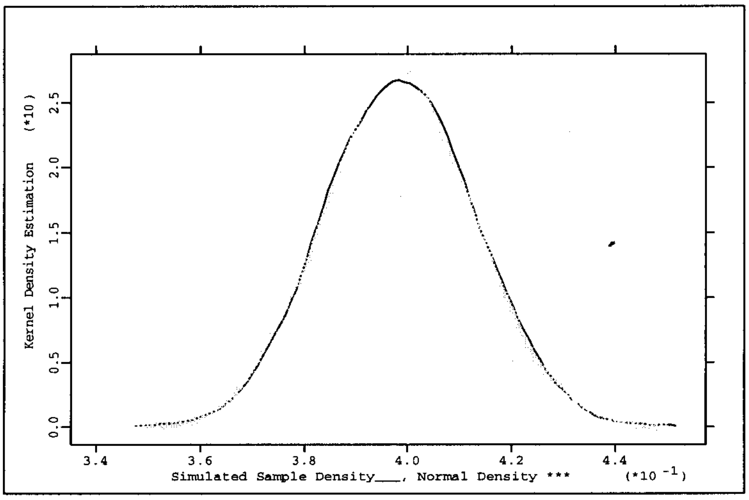


Figure 4: Density of $\widehat{d_\varepsilon}$ with 4500 observations

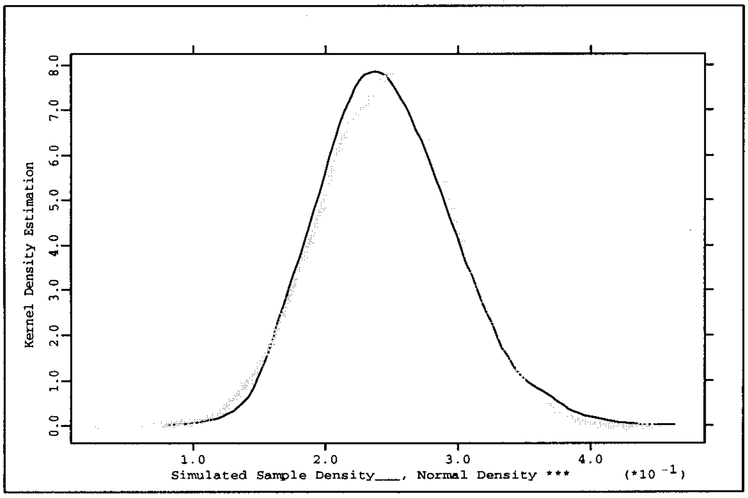


Figure 5: Density of $\widehat{d_\varepsilon}$ with 8500 observations

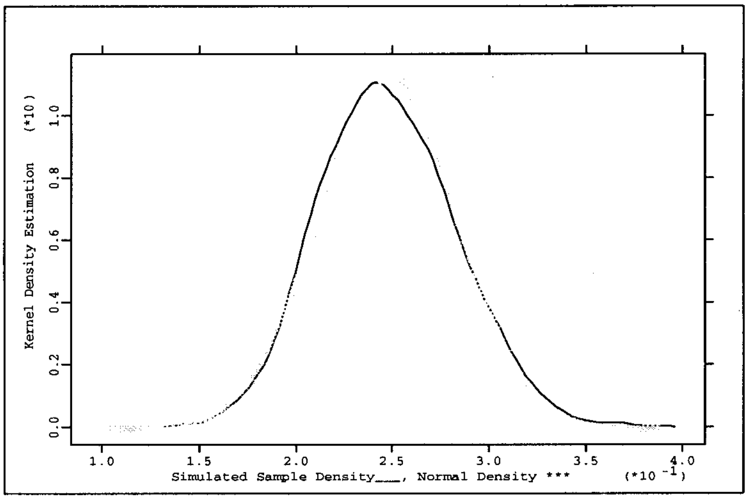


Figure 6: Density of $\widehat{\phi_1}$ with 4500 observations

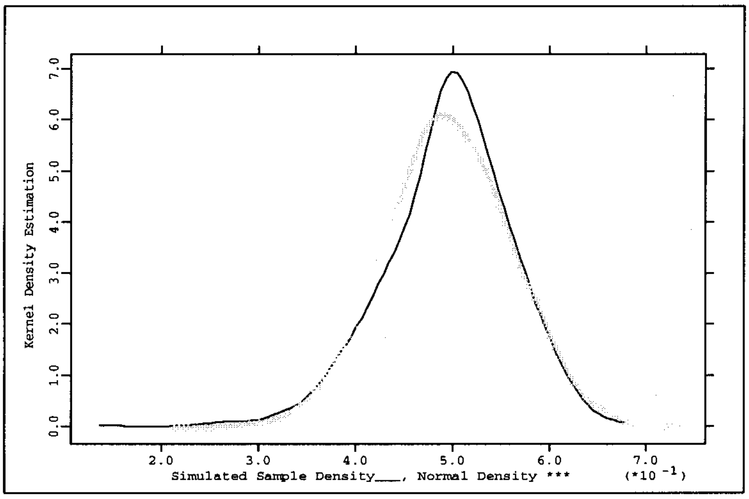


Figure 7: Density of $\widehat{\phi_1}$ with 8500 observations

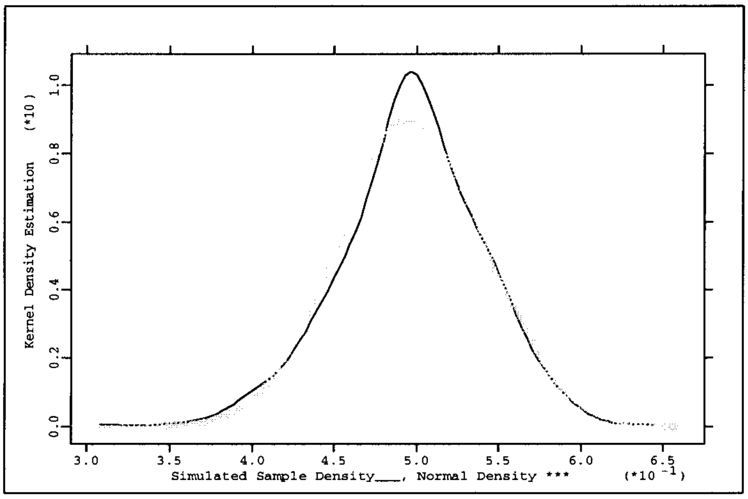


Figure 8: Kernel density estimate of the Ljung-Box statistic Q_{50} of an ARFIMA$(1, d_y, 1)$-FIGARCH $(1, d_\varepsilon, 1)$ under correct specification, with a simulated density of the χ^2_{48} random variable.

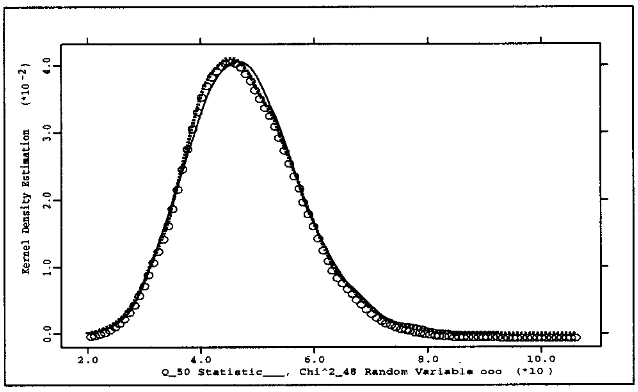


Figure 9: Kernel density estimate of the Ljung-Box statistic Q_{10} of an ARFIMA$(1, d_y, 1)$-FIGARCH $(1, d_\varepsilon, 1)$ under correct specification, with a simulated density of the χ^2_8 random variable.

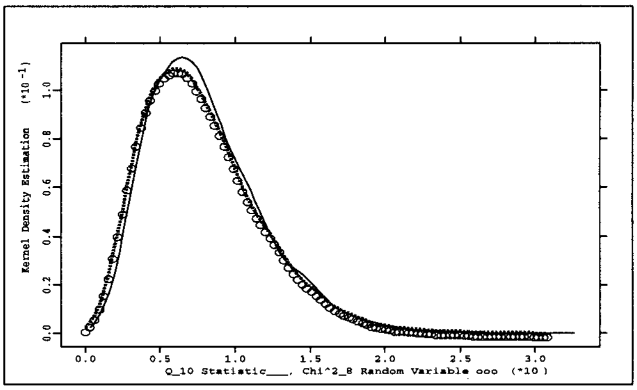


Figure 10: Kernel density estimate of the Ljung-Box statistic Q^2_{50} of an ARFIMA$(1, d_y, 1)$-FIGARCH $(1, d_\varepsilon, 1)$ under correct specification, with a simulated density of the χ^2_{48} random variable.

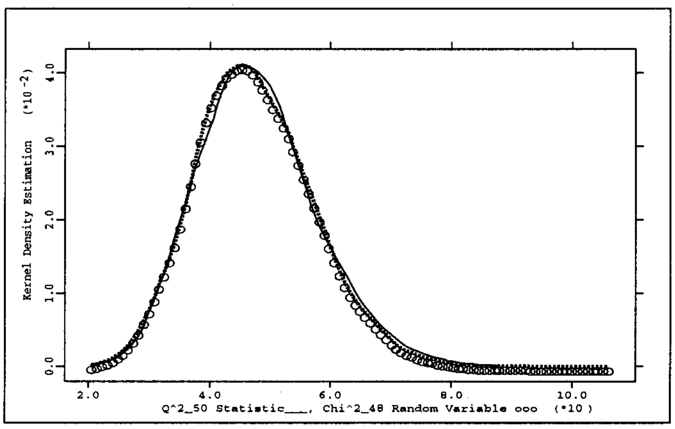


Figure 11: Kernel density estimate of the Ljung-Box statistic Q_{10}^2 of an ARFIMA$(1, d_y, 1)$-FIGARCH $(1, d_\varepsilon, 1)$ under correct specification, with a simulated density of the χ_8^2 random variable.

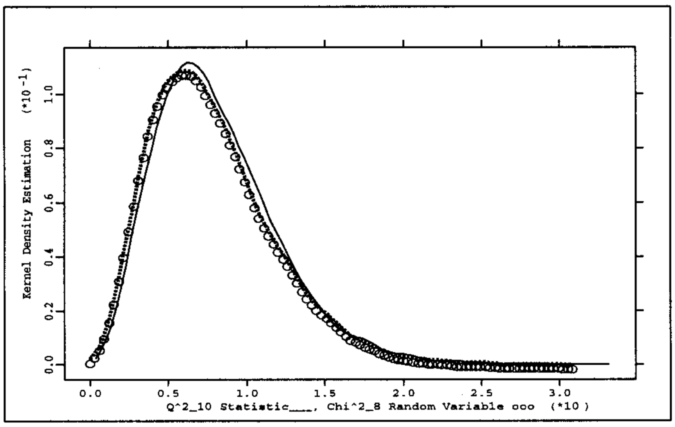


Figure 12: Kernel density estimate of the Ljung-Box Q_{10} statistic of an AR(2)-FIGARCH $(1, d_\varepsilon, 1)$ when the DGP is an ARFI$(1, d_y)$-FIGARCH $(1, d_\varepsilon, 1)$, for $d_y = 0.06$, with a simulated density of the χ^2_8 random variable.

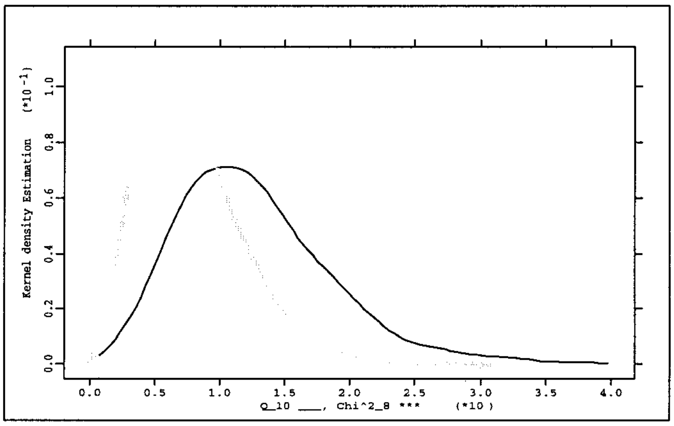
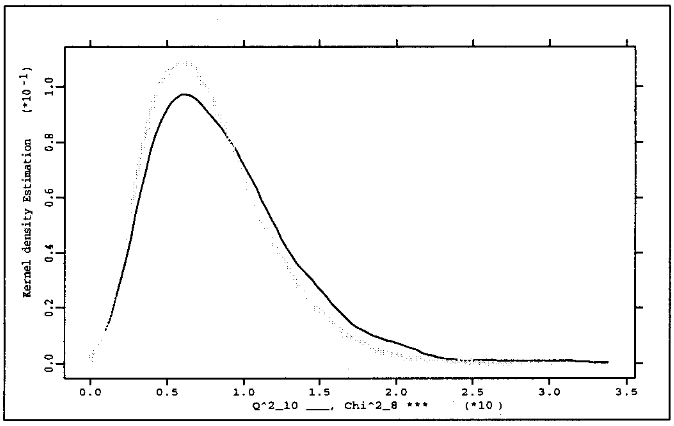


Figure 13: Kernel density estimate of the Ljung-Box Q^2_{10} statistic of an AR(2)-FIGARCH $(1, d_\varepsilon, 1)$ when the DGP is an ARFI$(1, d_y)$-FIGARCH $(1, d_\varepsilon, 1)$, for $d_y = 0.06$, with a simulated density of the χ^2_8 random variable.



$$\begin{pmatrix} (1-\psi_{1,1}L)(1-L)^{d_y}(y_{1,t}-\mu_1) \\ (1-\psi_{2,1}L)(1-L)^{d_y}(y_{2,t}-\mu_2) \\ (1-\psi_{1,1}L)(1-L)^{d_y}(y_{3,t}-\mu_3) \end{pmatrix} = \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

with

$$\begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s_{11,t} & s_{12,t} & s_{13,t} \\ s_{21,t} & s_{22,t} & s_{23,t} \\ s_{31,t} & s_{32,t} & s_{33,t} \end{pmatrix}\right]$$

$$
\begin{pmatrix} s_{11,t} \\ s_{22,t} \\ s_{33,t} \\ s_{12,t} \\ s_{13,t} \\ s_{23,t} \end{pmatrix} = \left(\begin{array}{c} \omega_{1,1} + \left(1 - (1-L)^{d_{1,1}}\right) \varepsilon_{1,t}^2 \\ \dfrac{\omega_{2,2}}{1-\beta_2(1)} + \left(1 - \dfrac{(1-\phi_2 L)(1-L)^{d_{2,2}}}{1-\beta_2 L}\right) \varepsilon_{2,t}^2 \\ \omega_{3,3} + \left(1 - (1-\phi_3 L)(1-L)^{d_{3,3}}\right) \varepsilon_{3,t}^2 \\ \omega_{1,2} + \left(1 - (1-L)^{d_{1,2}}\right) \varepsilon_{1,t}\varepsilon_{2,t} \\ \omega_{1,3} + \left(1 - (1-L)^{d_{1,3}}\right) \varepsilon_{1,t}\varepsilon_{3,t} \\ \omega_{2,3} + \left(1 - (1-L)^{d_{2,3}}\right) \varepsilon_{2,t}\varepsilon_{3,t} \end{array} \right.
$$

Table 9: Estimation results of the restricted trivariate double long-memory model. Robust standard error are between parentheses.

Parameters	Restricted model
μ_1	0.0139 (0.0037)
μ_2	-0.0051 (0.0027)
μ_3	0.0041 (0.0032)
$\psi_{1,1}$	-0.1036 (0.0100)
$\psi_{1,2}$	-0.1685 (0.0118)
$\psi_{1,3}$	$\psi_{1,1}$
d_y	-0.0618 (0.0072)
$\omega_{1,1}$	0.1835 (0.0180)
$\omega_{2,2}$	0.0562 (0.0137)
$\omega_{3,3}$	0.1223 (0.0117)
$\omega_{1,2}$	0.0580 (0.0062)
$\omega_{1,3}$	0.0935 (0.0081)
$\omega_{2,3}$	0.0657 (0.0066)
β_2	0.4908 (0.1139)
ϕ_2	0.5555 (0.1173)
ϕ_3	0.0529 (0.0198)
$d_{1,1}$	0.1626 (0.0114)
$d_{2,2}$	$d_{1,1}$
$d_{3,3}$	$d_{1,1}$
$d_{1,2}$	0.0600 (0.0082)
$d_{1,3}$	0.1038 (0.0065)
$d_{2,3}$	$d_{1,3}$
Log-likelihood	-34772.184

Table 8: Estimation results for the ARFIMA-FIGARCH on the series of absolute returns. (Robust t-statistics are given between parentheses, the percentiles of the χ^2 distribution, with adjusted number of degrees of freedom, are between squared brackets)

Parameters	ARFIMA-FIGARCH
μ_y	0.6162 (3.42)
d_y	0.5618 (9.42)
ψ_2	-0.1188 (-10.47)
ψ_4	0.8398 (24.30)
ψ_5	-0.1262 (-8.48)
θ_1	-0.5746 (-10.38)
θ_3	-0.0832 (-7.67)
θ_4	-0.8576 (-23.09)
θ_5	0.6129 (11.55)
ω	0.0141 (2.46)
d_ε	0.3399 (6.72)
β_1	0.5931 (6.36)
ϕ_1	0.2694 (3.38)
Q_{20}	12.47 [0.511]
Q_{50}	60.20 [0.962]
Q_{80}	91.94 [0.933]
Q_{100}	104.56 [0.806]
Q_{200}	202.34 [0.692]
Q^2_{20}	12.06 [0.156]
Q^2_{50}	45.41 [0.421]
Q^2_{80}	77.44 [0.503]
Q^2_{100}	96.97 [0.489]
Q^2_{200}	195.00 [0.453]

ARFIMA-FIGARCH MODEL

Bias and Consistency

Table 1: Estimation results of the ARFIMA$(1, d_y, 1)$-FIGARCH$(1, d_\varepsilon, 1)$ (i) with 10,000 simulations, (ii) and (iii) with 10,000 simulations.

Parameters		(i) Normal error terms		(ii) t_7 error terms		(iii) t_7 error terms with exact likelihood	
	Values	Mean	SE_{HCCM}	Mean	SE_{HCCM}	Mean	SE_{HCCM}
ψ_1	0.15	0.1515	0.0394	0.1533	0.047	0.1522	0.0406
μ_y	1.00	1.0066	0.4175	1.0088	0.410	1.0050	0.3809
d_y	0.40	0.3989	0.0192	0.3962	0.022	0.3986	0.0196
θ_1	0.40	0.3998	0.0259	0.4001	0.032	0.3993	0.0268
ω	0.10	0.1075	0.0163	0.1089	0.020	0.1067	0.0169
d_ε	0.25	0.2491	0.0478	0.2500	0.072	0.2504	0.0575
ϕ_1	0.50	0.4894	0.0632	0.4902	0.094	0.4829	0.0752
β_1	0.20	0.1885	0.0524	0.1906	0.077	0.1851	0.0634
η	—	—	—	—	—	7.1033	0.6902

Table 2: Estimation results of the ARFIMA$(0, d_y, 0)$-FIGARCH$(0, d_\varepsilon, 0)$ for samples of 4500 and 8500 observations, with the ratio of the estimated standard errors from the HCCM matrix (SE_{HCCM}), after 2000 simulations.

Parameters		4500 observations		8500 observations		$\frac{SE_{HCCM}(4000)}{SE_{HCCM}(8000)}$
	Values	Mean	SE_{HCCM}	Mean	SE_{HCCM}	Mean
μ_y	1.00	1.0063	0.3402	1.0059	0.2265	1.5020
d_y	0.40	0.4008	0.0131	0.4012	0.0089	1.4720
ω	0.10	0.1067	0.0231	0.1033	0.0148	1.5608
d_ε	0.50	0.5023	0.0262	0.5014	0.0174	1.5057

Heteroskedastic consistent covariance matrix (HCCM) estimator: $\mathrm{n}^{-1}\mathcal{H}^{-1}(\hat{\zeta})\,\mathcal{I}(\hat{\zeta})\,\mathcal{H}^{-1}(\hat{\zeta})$

Empirical distribution of the Ljung-Box statistic

Table 3: Empirical distributions of the Q_K, Q_K^A and Q_K^2 statistics for an ARFIMA$(1, d_y, 1)$-FIGARCH$(1, d_\varepsilon, 1)$, with 10,000 simulations

Percentiles	Q_{10}	Q_{10}^A	Q_{10}^2	Q_{50}	Q_{50}^A	Q_{50}^2
25%	0.2184	0.2194	0.2152	0.2417	0.2444	0.2430
50%	0.4789	0.4710	0.4668	0.4880	0.4876	0.4937
75%	0.7381	0.7308	0.7309	0.7423	0.7393	0.7492
95%	0.9503	0.9476	0.9500	0.9457	0.9454	0.9461

Table 4: Empirical distributions of the Q_K, Q_K^A and Q_K^2 statistics for an ARFIMA$(1, d_y, 0)$-FIGARCH$(1, d_\varepsilon, 1)$, with 2000 simulations

Percentiles	Q_{10}	Q_{10}^A	Q_{10}^2	Q_{50}	Q_{50}^A	Q_{50}^2
25%	0.2780	0.2175	0.2205	0.2700	0.2460	0.2460
50%	0.5290	0.4735	0.4715	0.5185	0.5010	0.5010
75%	0.7865	0.7470	0.7400	0.7650	0.7325	0.7495
95%	0.9550	0.9510	0.9530	0.9470	0.9445	0.9530

Table 5: Lo statistic: Frequencies of acceptation of the null hypothesis of no long-memory, upper tailed test, test size 5%. (with 10.000 simulations)

d	0	1	2	3	4	5	6	7	8	9
0.06	0.4112	0.5454	0.6000	0.6466	0.6767	0.6999	0.7177	0.7338	0.7437	0.7570
0.10	0.1762	0.1824	0.3540	0.4044	0.4471	0.4787	0.5074	0.5303	0.5464	0.5640
0.40	0.0000	0.0000	0.0001	0.0001	0.0002	0.0004	0.0005	0.0009	0.0015	0.0020

Table 6: KPSS statistic: Frequencies of acceptation of the null hypothesis of no long-memory, upper tailed test, test size 5%. (with 10.000 simulations)

d	0	1	2	3	4	5	6	7	8	9
0.06	0.6544	0.7187	0.7510	0.7706	0.7830	0.7948	0.8035	0.8110	0.8165	0.8218
0.10	0.4642	0.5422	0.5861	0.6166	0.6387	0.6564	0.6689	0.6801	0.6897	0.6975
0.40	0.0001	0.0020	0.0038	0.0070	0.0101	0.0136	0.0181	0.0230	0.0277	0.0327

Table 7: V/S statistic: Frequencies of acceptation of the null hypothesis of no long-memory, upper tailed test, test size 5%. (with 10.000 simulations)

d	0	1	2	3	4	5	6	7	8	9
0.06	0.5188	0.6090	0.6384	0.6886	0.7089	0.7212	0.7342	0.7443	0.7533	0.7614
0.10	0.2832	0.3791	0.4376	0.4798	0.5089	0.5331	0.5509	0.5682	0.5811	0.5944
0.40	0.0000	0.0000	0.0000	0.0002	0.0004	0.0011	0.0022	0.0036	0.0053	0.0064